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Robotic Dexterity via Nonholonomy

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In this paper we consider some new avenues that the design and control of versatile robotic end–effectors, or "hands", are taking to tackle the stringent requirements of both industrial and servicing applications. A point is made in favour of the so–called minimalist approach to design, consisting in the reduction of the hardware complexity to the bare minimum necessary to fulfill the specifications. It will be shown that to serve this purpose best, more advanced understanding of the mechanics and control of the hand–object system is necessary. Some advancements in this direction are reported, while few of the many problems still open are pointed out.

1. Introduction

The development of mechanical hands for grasping and fine manipulation of objects has been an important part of robotics research since its beginnings. Comparison of the amazing dexterity of the human hand with the extremely elementary functions performed by industrial grippers, compelled many robotics researchers to try and bring some of the versatility of the anthropomorphic model in robotic devices. From the relatively large effort spent by the research community towards this goal, several robot hands sprung out in laboratories all over the world. The reader is referred to detailed surveys such as e.g. [15, 34, 13, 27, 2].

Multifingered, "dextrous" robot hands often featured very advanced mechanical design, sensing and actuating systems, and also proposed interesting analysis and control problems, concerning e.g. the distribution of control action among several agents (fingers) subject to complex nonlinear bounds. Notwithstanding the fact that hands designed in that phase of research were often superb engineering projects, the community had to face a very poor penetration to the factory floor, or to any other scale application. Among the various reasons for this, there is undoubtedly the fact that dextrous robot hands were too mechanically complex to be industrially viable in terms of cost, weight, and reliability.

Reacting to this observation, several researchers started to reconsider the problem of obtaining good grasping and manipulation performance by using mechanically simpler devices. This approach can be seen as an embodiment of a more general, "minimalist" attitude at robotics design (see e.g. works reported in [3]). It often turns out that this is indeed possible, provided that more sophisticated analysis, programming and control tools are employed.

The challenge is to make available theoretical tools which allow to reduce the hardware cost at little incremental cost of basic research.

One instance of this process of hardware reduction without sacrificing performance can be seen in devices for "power grasping", or "whole-arm manipulation", i.e. devices that exploit all their parts to contact and constrain the manipulated part, and not just their end-effectors (or fingertips, in the case of hands). From the example of human grasp, it is evident that power grasps using also the palm and inner phalanges are more robust than fingertip grasping, for a given level of actuator strength. However, using inner parts of the kinematic chain, which have reduced mobility in their operational space, introduces important limitations in terms of controllability of forces and motions of the manipulated part, and ensue non-trivial complications in control. Such considerations are dealt with at some length in references [37, 36], and will not be reported here.

In this paper, we will focus on the achievement of dexterity with simplified hardware. By dexterity we mean here (in a somewhat restrictive sense) the ability of a hand to relocate and reorient an object being manipulated among its fingers, without loosing the grasp on it. Salisbury [23] showed first that the minimum theoretical number of d.o.f.'s to achieve dexterity in a hand with rigid, hard-finger, non-rolling and non-sliding contacts, is 9. As a simple explanation of this fact, consider that at least three hard-fingers are necessary to completely restrain an object. On the other hand, as no rolling nor sliding is allowed, fingers must move so as to track with the contact point on their fingertip the trajectory generated by the corresponding contact point on the object, while this moves in 3D space. Hence, 3 d.o.f.'s per finger are strictly necessary. If the non-rolling assumption is lifted, however, the situation changes dramatically, as nonholonomy enters the picture. The analysis of manipulation in the presence of rolling has been pioneered by Montana [25], Cai and Roth [9], Cole, Hauser, and Sastry [11], Li and Canny [20].

In this paper we report on some results that have been obtained in the study of rolling objects, in view of the realization of a robot gripper that exploits rolling to achieve dexterity. A first prototype of such device, achieving dexterity with only four actuators, was presented by Bicchi and Sorrentino [5]. Further developments have been described in [4, 22].

Although nonholonomy seems to be a promising approach to reducing the complexity, cost, weight, and unreliability of the hardware used in robotic hands, it is true in general that planning and controlling nonholonomic systems is more difficult than holonomic ones. Indeed, notwithstanding the efforts spent by applied mathematicians, control engineers, and roboticists on the subject, many open problems remain unsolved at the theoretical level, as well as at the computational and implementation level.

The rest of the paper is organized as follows. In Sect. 2. we overview applications of nonholonomic mechanical systems to robotics, and provide a rather broad definition of nonholonomy that allows to treat in a uniform way phenomena with a rather different appearance. In Sect. 3. we make the point on the state-of-the-art in manipulation by rolling, with regard to both regular and irregular surfaces. We conclude the paper in Sect. 4. with a discussion of the open problems in planning and controlling such devices.

2. Nonholonomy on Purpose

A knife-edge cutting a sheet of paper and a cat falling onto its feet are common examples of natural nonholonomic systems. On the other hand, bicycles and cars (possibly with trailers) are familiar examples of artificially designed nonholonomic devices. While nonholonomy in a system is often regarded as an annoying side-effect of other design considerations (this is how most people consider e.g. maneuvering their car for parking in parallel), it is possible that nonholonomy is introduced on purpose in the design in order to achieve specific goals. The Abdank-Abakanowicz's integraph and the Henrici-Corradi harmonic analyzer reported by Neimark and Fufaev [30] are nineteenth-century, very ingenuous examples in this sense, where the nonholonomy of rolling of wheels and spheres are exploited to mechanically construct the primitive and the Fourier series expansion of a plotted function, respectively.

Another positive aspect of nonholonomy, and actually the one that motivates the perspective on robotic design considered in this paper, is the reduction in the number of actuators it may allow. In order to make the idea evident, consider the standard definition of a nonholonomic system as given in most mechanics textbooks:

Definition 2.1. *A mechanical system described by its generalized coordinates* $\mathbf{q} = (q_1, q_2, \ldots, q_n)^T$ *is called nonholonomic if it is subject to constraints of the type*

$$\mathbf{c}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = 0, \tag{2.1}$$

and if there is no equation of the form $\hat{\mathbf{c}}(\mathbf{q}(t)) = 0$ *such that* $\frac{d\hat{\mathbf{c}}(\mathbf{q}(t))}{dt} = \mathbf{c}(\mathbf{q}(t), \dot{\mathbf{q}}(t))$. *If linear in* $\dot{\mathbf{q}}$, *i.e. if it can be written as*

$$\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = 0,$$

a constraint is called Pfaffian.

A Pfaffian set of constraints can be rewritten in terms of a basis $\mathbf{G}(q)$ of the kernel of $\mathbf{A}(q)$, as[1]

[1] in more precise geometrical terms, the rows of $A(q)$ are the covector fields of the active constraints forming a codistribution, and the columns of $\mathbf{G}(q)$ are a set of vector fields spanning the annihilator of the constraint codistribution. If the constraints are smooth and independent, both the codistribution and distribution are nonsingular.

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q})\mathbf{u} \tag{2.2}$$

This is the standard form of a nonlinear, driftless control system. In the related vocabulary, components of $\mathbf{u}$ are "inputs". The non–integrability of the original constraint has its control–theoretic counterpart in Frobenius Theorem, stating that a nonsingular distribution is integrable if and only if it is involutive. In other words, if the distribution spanned by $\mathbf{G}(q)$ is not involutive, motions along directions that are not in the span of the original vector fields are possible for the system.

From this fact follows the most notable characteristic of nonholonomic systems with respect to minimalist robotic design, i.e. that they can be driven to a desired equilibrium configuration in a d–dimensional configuration manifold using less than d inputs. In a kinematic bicycle, for instance, two inputs (the forward velocity and the steering rate) are enough to steer the system to any desired configuration in its 4–dimensional state space. Notice that these "savings" are unique to nonlinear systems, as a linear system always requires as many inputs as states to be steered to arbitrary equilibrium states (this property being in fact equivalent to functional controllability of outputs for linear systems).

Since "inputs" in engineering terms translates into "actuators", devices designed by intentionally introducing nonholonomic mechanisms can spare hardware costs without sacrificing dexterity. Few recent works in mechanism design and robotics reported on the possibility of exploiting nonholonomic mechanical phenomena in order to design devices that achieve complex tasks with a reduced number of actuators (see e.g. [39, 5, 12, 35]).

It is worthwhile mentioning at this point that nonholonomy occurs not only because of rolling, but also in systems of different types, such as for instance:

- Systems subject to conservation of angular momentum, as is the case of the falling cat. This type of nonholonomy can be exploited for instance for orienting a satellite with only two torque actuators [26], or reconfiguring a satellite–manipulator system [29, 17].
- Underactuated mechanical systems, such as robot arms with some free joints, usually result in dynamic, second–order nonintegrable, nonholonomic constraints [32]. This may allow reconfiguration of the whole system by controlling only actuated joints, as e.g. in [1, 12].
- Nonholonomy may be exhibited by piecewise holonomic systems, such as switching electrical systems [19], or mechanical systems with discontinuous phenomena due to intermittent contacts, Coulomb friction, etc.. Brockett [8] discussed some deep mathematical aspects of the rectification of vibratory motion in connection with the problem of realizing miniature piezoelectric motors (see Fig. 2.1). He stated in that context that "from the point of view of classical mechanics, rectifiers are necessarily non–holonomic systems". Lynch and Mason [21] used controlled slippage to build a 1–joint "manipulator" that can reorient and displace arbitrarily most planar mechanical parts on a a conveyor belt, thus achieving control on a 3 dimensional configuration space by using one controlled input (the manipulator's actuator) and one constant drift vector (the belt velocity). Ostrowski and Burdick [33] gave a rather general mathematical model of locomotion in natural and artificial systems, showing how basically any locomotion system is a nonholonomic system. In these examples, however, a more general definition of nonholonomy has to be considered to account for the discontinuous nature of the phenomena occurring.
- Nonholonomy can be exhibited by inherently discrete systems. The simple experiment of rolling a die onto a plane without slipping, and bringing it back after any sufficiently rich path, shows that its orientation has changed in general (see Fig. 2.2). The fact that almost all polyhedra can be brought close to a desired position and orientation by rolling on a plate, to be discussed shortly, can be used to build dextrous hands for manipulation of general (non–smooth) mechanical parts. Once again, these nonholonomic phenomena can not be described and studied based on classical differential geometric tools.

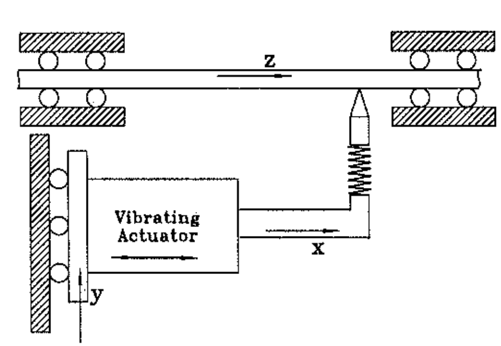

Fig. 2.1. Illustrating the principle of a mechanical rectifier after Brockett. The tip of the vibrating element oscillates in the x direction, while a variable pressure against the rod is controlled in the y direction. When the contact pressure is larger than a threshold y_0, dry friction forces the rod to translate in the z direction

A more general definition than (2.1) is given below for time–invariant systems:

Definition 2.2. *Consider a system evolving in a configuration space $\mathcal{Q}$, a time set (continuous or discrete) $\mathcal{T}$, and a bundle of input sets $\mathcal{A}$, such that for each input set $\mathcal{A}(\mathbf{q}, \mathbf{t})$ defined at $\mathbf{q} \in \mathcal{Q}$, $\mathbf{t} \in \mathcal{T}$, it holds* $\mathbf{a} : (\mathbf{q}, \mathbf{t}) \to \mathbf{q}'$, $\mathbf{q}' \in \mathcal{Q}, \forall \mathbf{a} \in \mathcal{A}(\mathbf{q}, \mathbf{t})$. *If it is possible to decompose $\mathcal{Q}$ in a projection or* base *space $\mathcal{B} = \Pi(\mathcal{Q})$ and a* fiber bundle *$\mathcal{F}$, such that $\mathcal{B} \times \mathcal{F} = \mathcal{Q}$ and there exists a sequence of inputs in $\mathcal{A}$ starting at $\mathbf{q}_0$ and steering the* *system to* $\mathbf{q}^* = \mathbf{a}_n(\mathbf{q}_{n-1}, \mathbf{t}_{n-1}) \circ \cdots \circ \mathbf{a}_1(\mathbf{q}_0, \mathbf{t}_0)$, *such that* $\Pi(\mathbf{q}_0) = \Pi(\mathbf{q}^*)$ *but* $\mathbf{q}_0 \neq \mathbf{q}^*$, *then the system is nonholonomic at* $\mathbf{q}_0$.

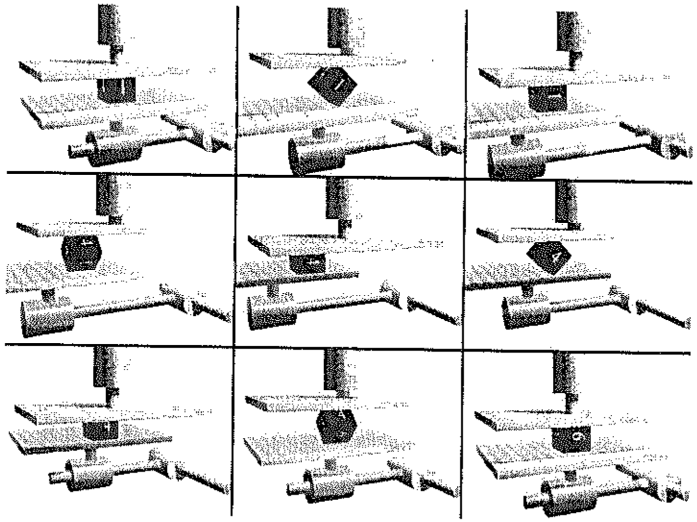

Fig. 2.2. A die being rolled between two parallel plates. After four tumbles over its edges, the center of the die comes back to its initial position, while its orientation has changed

According to this definition, a system is nonholonomic if there exist controls that make some configurations go through closed cycles, while the rest of configurations undergo net changes per cycle (see Fig. 2.3).

For instance, in the continuous, nonholonomic Heisenberg system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -x_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ x_1 \end{bmatrix}, \qquad (2.3)$$

it is well known (see e.g. [8]) that "Lie–bracket motions" in the direction of

$$[\mathbf{G}_1(\mathbf{x}), \mathbf{G}_2(\mathbf{x})] = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

are generated by any pair of simultaneous periodic zero–average functions $u_1(\cdot), u_2(\cdot)$. Definition 1 specializes in this case with $\mathcal{Q} = \mathbb{R}^3$, $\mathcal{T} = \mathbb{R}_+$, and

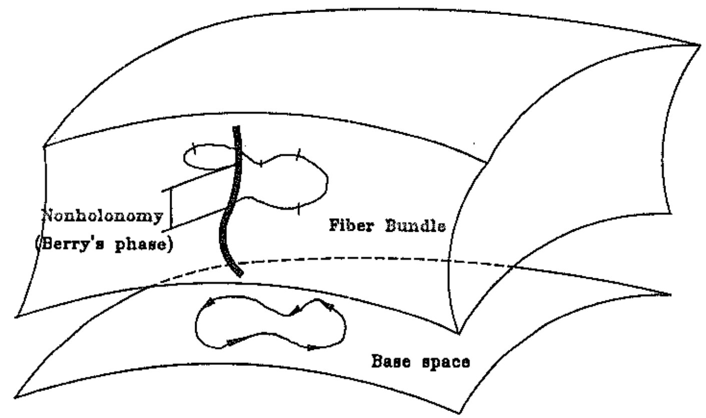

Fig. 2.3. Illustrating the definition of nonholonomic systems

$\mathcal{A}(\mathbf{x}, t) = \{\exp(t(\mathbf{G}_1 u_1 + \mathbf{G}_2 u_2))\,\mathbf{x}, \forall$ piecewise continuous $u_i(\cdot) : [0, t] \to \mathbb{R}, i = 1, 2.\}$. The base space is simply the x_1, x_2 plane, and the fibers are in the x_3 direction. Periodic inputs generate closed paths in the base space, corresponding to a fiber motion of twice the (signed) area enclosed on the base by the path.

As an instance of embodiment of the above definition in a piecewise holonomic system, consider the simplified version of one of Brockett's rectifiers in Fig. 2.1. The two regimes of motion, without and with friction, are

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2, \quad y < y_0;$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2, \quad y \geq y_0,$$

respectively. In this case, base variables can be identified as x and y, while the fiber variable is z. Time is continuous, but the input bundle is discontinuous:

$$\mathcal{A}(x,y,z,t) = \begin{cases} y < y_0 : & \begin{cases} x & \rightarrow & x + \int_0^t u_1(\sigma)d\sigma; \\ y & \rightarrow & y + \int_0^t u_2(\sigma)d\sigma; \\ z & \rightarrow & z; \end{cases} \\ y \geq y_0 : & \begin{cases} x & \rightarrow & x + \int_0^t u_1(\sigma)d\sigma; \\ y & \rightarrow & y + \int_0^t u_2(\sigma)d\sigma; \\ z & \rightarrow & z + \int_0^t u_1(\sigma)d\sigma. \end{cases} \end{cases}$$

By changing frequency and phase of the two inputs, different directions and velocities of the rod motion can be achieved. Note in particular that input u_1 need not actually to be tuned finely, as long as it is periodic, and can be fixed e.g. as a resonant mode of the vibrating actuator. Fixing a periodic $u_1(\cdot)$ and tuning only u_2 still guarantees in this case the (non-local) controllability of the nonholonomic system: notice here the interesting connection with results on controllability of systems with drift reported by Brockett ([6], Theorem 4 and Hirschorn's Theorem 5).

Finally, consider how the above definition of nonholonomic system specializes to the case of rolling a polyhedron. Considering only configurations with one face of the polyhedron sitting on the plate, these can be described by fixing a point and a line on the polyhedron (excluding lines that are perpendicular to any face), taking their normal projections to the plate, and affixing coordinates x, y to the projected point, and θ to the angle of the projected line, with respect to some reference frame fixed to the plate. Therefore, $\mathcal{Q} = \mathbb{R}^2 \times S^1 \times F$, where F is the finite set of m face of the polyhedron. As the only actions that can be taken on the polyhedron are assumed to be "tumbles", i.e. rigid rotations about one of edges of the face currently lying on the plate that take the corresponding adjacent faces down to the plate, we take $\mathcal{T} = \mathbb{N}_+$ and $\mathcal{A}$ the bundle of m different, finite sets of neighbouring configurations just described. Figure 2.2 shows how a closed path in the base variables (x, y) generates a $\pi/2$ counterclockwise rotation and a change of contact face.

3. Systems of Rolling Bodies

For the reader's convenience, we report here some preliminaries that help in fixing the notation and resume the background. For more details, see e.g. [28, 5, 4, 10].

3.1 Regular Surfaces

The kinematic equations of motion of the contact points between two bodies with regular surface (i.e. with no edges or cusps) rolling on top of each other describe the evolution of the (local) coordinates of the contact point on the finger surface, $\alpha_f \in \mathbb{R}^2$, and on the object surface, $\alpha_o \in \mathbb{R}^2$, along with the holonomy angle ψ between the x–axes of two gauss frames fixed on the surfaces at the contact points, as they change according to the rigid relative motion of the finger and the object described by the relative velocity $\mathbf{v}$ and angular velocity ω. According to the derivation of Montana [25], in the presence of friction one has

$$\begin{aligned} \dot{\alpha}_f &= \mathbf{M}_f^{-1}\mathbf{K}_r^{-1} \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix}; \\ \dot{\alpha}_o &= \mathbf{M}_o^{-1}\mathbf{R}_\psi\mathbf{K}_r^{-1} \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix}; \\ \dot{\psi} &= \mathbf{T}_f\mathbf{M}_f\dot{\alpha}_f + \mathbf{T}_o\mathbf{M}_o\dot{\alpha}_o; \end{aligned} \tag{3.1}$$

where $\mathbf{K}_r = \mathbf{K}_f + \mathbf{R}_\psi\mathbf{K}_o\mathbf{R}_\psi$ is the relative curvature form, $\mathbf{M}_o, \mathbf{M}_f, \mathbf{T}_o, \mathbf{T}_f$ are the object and finger metric and torsion forms, respectively, and

$$\mathbf{R}_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{bmatrix}.$$

The rolling kinematics (3.1) can readily be written, upon specialization of the object surfaces, in the standard control form

$$\dot{\xi} = \mathbf{g}_1(\xi)v_1 + \mathbf{g}_2(\xi)v_2, \tag{3.2}$$

where the state vector $\xi \in \mathbb{R}^5$ represents a local parameterization of the configuration manifold, and the system inputs are taken as the relative angular velocities $v_1 = \omega_x$ and $v_2 = \omega_y$. Applying known results from nonlinear system theory, some interesting properties of rolling pairs have been shown. The first two concern controllability of the system:

Theorem 3.1. *(from [20]) A kinematic system comprised of a sphere rolling on a plane is completely controllable. The same holds for a sphere rolling on another sphere, provided that the radii are different and neither is zero.*

Theorem 3.2. *(from [4]) A kinematic system comprised of any smooth, strictly convex surface of revolution rolling on a plane is completely controllable.*

Remark 3.1. Motivated by the above results, it seems reasonable to conjecture that a kinematic system comprised of *almost* any pair of surfaces is controllable. Such fact is indeed important in order to guarantee the possibility of building a dextrous hand manipulating arbitrary (up to practical constraints) objects.

The following propositions concern the possibility of finding coordinate transforms and static state feedback laws which put the plate–ball system in special forms, which are of interest for designing planning and control algorithms:

Proposition 3.1. *The plate-ball system can not be put in* chained form *[27]; it is not* differentially flat *[38]; it is not* nilpotent *[14].*

These results prevent the few powerful planning and control algorithms known in the literature to be applied to kinematic rolling systems (of which the plate–ball system is a prototype). The following positive result however holds:

Theorem 3.3. *(from [5]). Assuming that either surface in contact is (locally) a plane, there exist a state diffeomorphism and a regular static state feedback law such that the kinematic equations of contact (3.1) assume a strictly triangular structure.*

The relevance of the strictly triangular form to planning stems from the fact that the flow of the describing ODE can be integrated directly by quadratures. Whenever it is possible to compute the integrals symbolically, the planning problem is reduced to the solution of a set of nonlinear algebraic equations, to which problem many well-known numerical methods apply.

3.2 Polyhedral Objects

The above mentioned simple experiment of rolling a die onto a plane without slipping hints to the fact that manipulation of parts with non–smooth (e.g. polyhedral) surface can be advantageously performed by rolling. However, while for analysing rolling of regular surfaces the powerful tools of differential geometry and nonlinear control theory are readily available, the surface regularity assumption is rarely verified with industrial parts, which often have edges and vertices.

Although some aspects of graspless manipulation of polyhedral objects by rolling have been already considered in the robotics literature, a complete study on the analysis, planning, and control of rolling manipulation for polyhedral parts is far from being available, and indeed it comprehends many aspects, some of which appear to be non–trivial. In particular, the lack of a differentiable structure on the configuration space of a rolling polyhedron deprives us of most techniques used with regular surfaces. Moreover, peculiar phenomena may happen with polyhedra, which have no direct counterpart with regular objects. For instance, in the examples reported in Figs. 3.1 and 3.2, it is shown that two apparently similar objects can reach configurations belonging to a very fine and to a coarse grid, respectively. In the second case, the mesh of the grid can actually be made arbitrarily small by manipulating the object long enough; in such case, the reachable set is said to be dense.

In fact, considering the description of the configuration set of a rolling polyhedron provided in Sect. 2. it can be observed that the state space $\mathcal{Q}$ is the union of l copies of $\mathbb{R}^2 \times S^1$. The subset of reachable configurations from some initial configuration $\mathcal{R}$ is given by the set of points reached by applying all admissible sequences of tumbles to the initial configuration. Notice that the set of all sequences is an infinite but countable set while the configuration space is a finite disjoint union of copies of a 3–dimensional variety. Thus, the set of reachable points is itself countable. Therefore, instead of the more familiar concept of "complete reachability" (corresponding to $\mathcal{R} = \mathcal{Q}$), it will only make sense to investigate a property of "dense reachability" defined as $\mathsf{closure}(\mathcal{R}) = \mathcal{Q}$. In other words, rolling a polyhedron on a plane has the dense reachability property if, for any configuration of the polyhedron and every $\epsilon \in \mathbb{R}_+$, there exists a finite sequence of tumbles that brings the polyhedron closer to the desired configuration than ϵ. We refer in particular to a distance on $\mathcal{Q}$ defined as

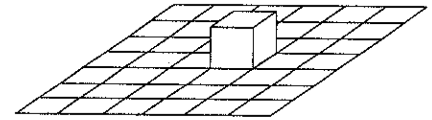

Fig. 3.1. A polyhedron whose reachable set is nowhere dense

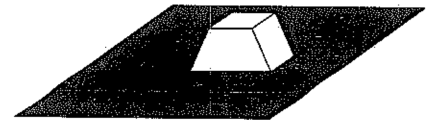

Fig. 3.2. A polyhedron whose reachable set is everywhere dense

$$\|(x_1, y_1, \theta_1, F_i) - (x_2, y_2, \theta_2, F_j)\| =$$
$$\max\left\{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},\ |\theta_1 - \theta_2|,\ 1 - \delta(F_i, F_j)\right\}.$$

The term *discrete* will be used for the negation of *dense*. On this regard, the following results were reported in [22] (we recall that the defect angle is 2π minus the sum of the planar angles of all faces concurring at that vertex, and equals the gaussian curvature that can be thought to be concentrated at the vertex):

Theorem 3.4. *The set of configurations reachable by a polyhedron is dense in $\mathcal{Q}$ if and only if there exists a vertex V_i whose defect angle is irrational with π.*

Theorem 3.5. *The reachable set is discrete in both positions and orientations if and only if either of these conditions hold:*

i) all angles of all faces (hence all defect angles) are integer multiples of $\pi/3$, and all lengths of the edges are rational w.r.t. each other;

ii) all angles of all faces (hence all defect angles) are $\pi/2$, and all lengths of the edges are rational w.r.t. each other;
iii) all defect angles are π.

Theorem 3.6. *The reachable set is dense in positions and discrete in orientations if and only if the defect angles are all rational w.r.t. π, and neither conditions i), ii), or iii) of Theorem 3.5 apply.*

Remark 3.2. Parts with a discrete reachable set are very special. Polyhedra satisfying condition i) of Theorem 3.2 are rectangular parallelepipeds, as e.g. a cube or a sum of cubes which is convex. Polyhedra as in condition ii) are those whose surface can be covered by a tessellation of equilateral triangles, as e.g. any Platonic solid except the dodecahedron. Condition iii) is only verified by tetrahedra with all faces equal.

Remark 3.3. Observe that in the above reachability theorems the conditions upon which the density or discreteness of the reachable set depends are in terms of rationality of certain parameters and their ratios. This entails that two very similar polyhedra may have qualitatively different reachable sets. This is for instance the case of the cube and truncated pyramid reported above in Figs. 3.1 and 3.2, respectively, where the latter can be regarded as obtained from the cube by slightly shrinking its upper face. In fact, for any polyhedron whose reachable set has a discrete structure, there exists an arbitrarily small perturbation of some of its geometric parameters that achieves density.

In view of these remarks, and considering that in applications the geometric parameters of the parts will only be known to within some *tolerance*, i.e. a bounded neighborhood of their nominal value, a formulation of the planning problems ignoring robustness of results w.r.t. modeling errors will make little sense in applications.

4. Discussion and Open Problems

One way of reducing what is probably the single highest cost source in robotic devices, i.e. their actuators, is offered by nonholonomy. It has been shown in this paper how nonholonomic phenomena are actually much more pervasive in practical applications than usually recognized. However, the real challenge posed by nonholonomic systems is their effective control, including analysis of their structural properties, planning, and stabilization. The situation of research in these fields is briefly reviewed below.

Controllability. A nonholonomic system according to the above definition may not be *completely* nonholonomic, i.e. not completely controllable in some or all of the various senses that are defined in the nonlinear control

literature. Detecting controllability is a much easier task for continuous driftless systems, such as e.g. the case of two bodies rolling on top of each other (see Eq. 3.1), because of the tools made available by nonlinear geometric control theory [16, 31]. Even in this case, though, there remains an open question to prove the conjecture that almost any pair of rolling bodies are controllable, or in other words, to characterize precisely the class of bodies which are not controllable, and to show that this subset is meager. Another question, practically a most important one, is to define a viable (i.e. computable and accurate) definition of a "controllability function" for nonholonomic systems, capable of conveying a sense of how intense the control activity has to be to achieve the manipulation goals, in a similar way as "manipulability" indices are defined in holonomic robots.
The controllability question is much harder for discontinuous systems or for systems with discrete input sets. As discussed above, relatively novel problems appear in the study of the reachable set, such as density or lattice structures. Very few tools are available from systems and automata theory to deal with such systems: consider to this regard that even the apparently simple problem of deciding the density of the reachable set of a 1–dimensional, linear problem

$$x_{k+1} = \lambda x_k + u_k, \quad u_k \in \text{ U, a finite set}$$

is unsolved to the best of our knowledge, and apparently not trivial in general. It is often useful in these problems to notice a possible group structure in the fiber motions induced by closed base space motions (see Fig. 2.3): such group analysis was actually instrumental to the results obtained for the polyhedron rolling problem.

Planning. The planning problem (i.e. the open–loop control) for some particular classes of nonholonomic systems is rather well understood. For instance, two–inputs nilpotentiable systems that can be put, by feedback transformation, in the so–called "chained" form, can be steered using sinusoids [28]; systems that are "differentially flat" can be planned looking at their (flat) outputs only [38]; systems that admit an exact sampled model (and maintain controllability under sampling) can be steered using "multirate control" [24]; nilpotent systems can be steered using the "constructive method" of [18]. However, as already pointed out, systems of rolling bodies do not fall into any of these classes. At present, planning motions of a spherical object onto a planar finger can be done in closed form, while for general objects only iterative solutions are available (e.g. the one proposed in [40]).

Stabilization. The control problem is particularly challenging for nonholonomic systems, due to a theorem of Brockett [7] that bars the possibility of stabilizing a nonholonomic vehicle about a nonsingular configuration by any continuous time–invariant static feedback. Non–smooth, time–varying, and dynamic extension algorithms have been proposed to face the point–stabilization problem for some classes of systems (e.g. chained–form). A stabilization method for a system of rolling bodies, or even for a sphere rolling on a planar finger, is not known to the authors.

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