Course on Model Predictive Control
Part IV – Nonlinear Model Predictive Control and Moving Horizon Estimation

Gabriele Pannocchia

Department of Chemical Engineering, **University of Pisa**, Italy
Email: `g.pannocchia@diccism.unipi.it`

Università di Pisa.
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Outline

1. Nonlinear Model Predictive Control: basics
   - General formulation and stability requirements
   - Examples of nonlinear MPC formulations
   - Suboptimal nonlinear MPC
   - Offset-free nonlinear MPC

2. Nonlinear Model Predictive Control: robustness
   - Inherent robustness
   - Robust tube-based nonlinear MPC

3. Moving horizon estimation
   - Introduction and full information estimator
   - Linear state estimation as an optimal control problem
   - Moving horizon estimation
   - Including constraints in the estimator

NMPC: dynamics, constraints and cost function

Nonlinear models

- Often **nonlinear models** are available in **continuous time**:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

- For **nonlinear MPC** design, we need a **discrete-time** model:

$$x(k+1) = F(x(k), u(k))$$
$$y(k) = h(x(k), u(k))$$

- Notice that: $F(x(k), u(k)) = x(k) + \int_{t_k}^{t_{k+1}} f(x, u)dt$
- For **simplicity**, we use the **notation**: $x^+ = f(x, u)$

Constraints and cost function

- **State and input** constraints: $x(k) \in \mathbb{X}, u(k) \in \mathbb{U}$
- **Stage** cost and **overall cost**: $V_N(x, \mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) + V_f(x(N))$

MPC: optimal control problem and assumptions

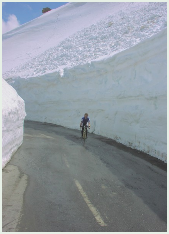
Main assumptions

- $\ell(\cdot)$ and $V_f(\cdot)$ are **positive definite**, and $\ell(0,0) = 0$, $V_f(0) = 0$
  $f(\cdot)$ is **continuous** and $f(0,0) = 0$
- **Control-invariant set** $\mathbb{X}_f \subseteq \mathbb{X}$: For any $x \in \mathbb{X}_f$, there exists $u \in \mathbb{U}$ such that: $V_f(f(x,u)) - V_f(x) \leq -\ell(x,u)$

Optimal control problem

Given the **current state** x, solve:

$$
\begin{aligned}
\mathbb{P}_N(x): \quad & \min_{\mathbf{u}} V_N(x,\mathbf{u}) \quad \text{s.t.} \\
& x^+ = f(x,u) \\
x(j) \in \mathbb{X} \quad & \text{for all } j = 0,\ldots,N-1 \\
u(j) \in \mathbb{U} \quad & \text{for all } j = 0,\ldots,N-1 \\
& x(N) \in \mathbb{X}_f
\end{aligned}
$$



NMPC: a note on computational aspects

General aspects

- The OCP is a **non-convex**, **nonlinear program**:
  - Computing $f(x, u)$ requires **ODE integration**
  - Finding **global optimum** is difficult
  - Solution algorithms are **time consuming**

Efficient NMPC methods [Diehl et al., 2008]

- Problem formulation aspects:
  - **Sequential**: **eliminate** the **state sequence** and solve for $\mathbf{u}$
  - **Simultaneous**: solve for both **state** and **input sequences** (multiple shooting, collocation methods, etc.)
- NLP methods:
  - **Sequential Quadratic Programming**: repeated **linearization** of constraints and **quadratic** expansion of the cost function
  - **Interior Point Methods**: direct solution of the (slightly modified) nonlinear optimality KKT conditions

NMPC: stability analysis

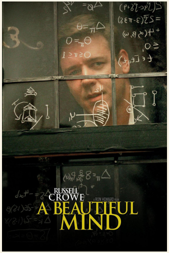
Lemma. Optimal cost decrease

Let $\kappa_N(x)$ denote the first element of the optimal control sequence $\mathbf{u}^0(x)$. For all $x \in \mathcal{X}_N$, there holds: $V_N^0(f(x,\kappa_N(x)) - V_N^0(x) \leq -\ell(x,\kappa_N(x))$

Proof

- Consider the **optimal input and state sequences** $\mathbf{u}^0(x) = \{u^0(0;x),\ldots,u^0(N-1;x)\}$, $\mathbf{x}^0(x) = \{x^0(0),\ldots,x^0(N)\}$
- At **next time**, given $x^+ = f(x,\kappa_N(x))$, consider a **candidate sequence** $\tilde{\mathbf{u}} := \{u^0(1;x),\ldots,u^0(N-1;x),u(N)\}$
- **Choose** $u(N) \in \mathbb{U}$ such that $x(N+1) = f(x^0(N;x),u(N)) \in \mathbb{X}_f$ and $V_f(x(N+1)) + \ell(x(N),u(N)) \leq V_f(x^0(N))$
- $\tilde{\mathbf{u}}$ is **feasible** and $V_N(x^+,\tilde{\mathbf{u}}) \leq V_N^0(x) - \ell(x,\kappa_N(x))$
- But **not optimal** for $\mathbb{P}_N(x^+)$. Thus:

$$V_N^0(x^+) \leq V_N(x^+,\tilde{\mathbf{u}}) \leq V_N^0(x) - \ell(x,\kappa_N(x))$$



NMPC: examples of different terminal constraints/costs

The earliest stable formulations [Mayne and Michalska, 1990, Michalska and Mayne, 1993]

- Terminal constraint "set" is the **origin**
  - (No) **Terminal cost**: $V_f(x) = 0$
  - **Terminal set**: $\mathbb{X}_f = \{0\}$
- **Dual-mode** formulation:
  - **Preliminary** operations ((A, B), linearized system matrices)
    - ★ Choose any K s.t. $A_K = A + BK$ is stable
    - ★ Set $Q^* = Q + K'RK$ and solve $P = A_K' P A_K + \mathbf{2}Q^*$.
    - ★ Define $\mathbb{X}_f = \{x \in \mathbb{R}^n \mid x'Px \leq \alpha$ is invariant for $x^+ = f(x, Kx)$
  - Mode 1: if $x \notin \mathbb{X}_f$ solve $\mathbb{P}_N(x)$ with $V_f = 0$
  - Mode 2: if $x \in \mathbb{X}_f$ use $u = Kx$

Quasi-infinite horizon [Chen and Allgower, 1998]

- Terminal **cost**: $V_f(x) = x'Px$
- Terminal **set**: $\mathbb{X}_f = \{x \in \mathbb{R}^n \mid x'Px \leq \alpha\}$, invariant for $x^+ = f(x, Kx)$

NMPC: omitting the terminal constraint

Is a terminal constraint set necessary?

- The addition of $V_f(\cdot)$ **does not affect materially** the OCP
- The addition of $x(N) \in \mathbb{X}_f$ **does**
- Is there an **implicit** way of **enforcing the constraint**?

Inflating the terminal penalty [Limon et al., 2006]

- Basic idea: **increase** $V_f(\cdot)$ enough to make $x(N) \in \mathbb{X}_f$ **inherently satisfied**
- **Modified cost function**, given $\beta > 1$

$$V_N^{\beta}(x, \mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) + \beta V_f(x(N))$$

- Let $\mathcal{X}_N = \{x \in \mathbb{R}^n \mid V_N^0(x) \leq \bar{V}\}$ and $\mathbb{X}_f = \{x \in \mathbb{R}^n \mid x' P x \leq \alpha\}$. **Choose any** $\beta \geq \bar{V}/\alpha$

Suboptimal nonlinear MPC

A neat suboptimal MPC framework [Scokaert et al., 1999]

- Given **current state** x, **previous control** sequence $\mathbf{u}^{-} = \{u^{-}(0), u^{-}(1), \ldots, u^{-}(N-1)\}$ and **state** sequence $\mathbf{x}^{-} = \{x^{-}(0), x^{-}(1), \ldots, x^{-}(N)\}$
- Build a **warm-start**: $\mathbf{u}_0 = \{u^{-}(1), \ldots, u^{-}(N-1), \kappa_f(x^{-}(N))\}$
- Perform **some iterations** to **improve** the **warm start**:

$$V_N(x, \mathbf{u}) \leq V_N(x, \mathbf{u}_0)$$

Take home message

- **Stability holds** for suboptimal MPC
- It is always a good idea to **warm start** nonlinear MPC solvers

Offset-free nonlinear MPC design

Augmented nonlinear system [Morari and Maeder, 2012]

- As in the linear case, **augment** the nominal system with **integrating disturbances**

$$x^+ = f_{aug}(x,u,d)$$
$$d^+ = d$$
$$y = h_{aug}(x,u,d)$$

- **Estimate** both (x,d) given the measurement of y

Target problem and deviation variables

- Given $\hat{d}$ solve a **target problem** to obtain (x_s,u_s)

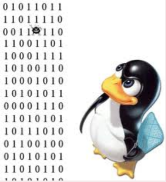
$$x_s = f_{aug}(x_s,u_s,\hat{d}), \quad u_s \in \mathbb{U}, \quad x_s \in \mathbb{X}$$

- **Deviation variables**, $\tilde{x} = x - x_s$, $\tilde{u} = u - u_s$, are **regulated to zero**

Inherent robustness of nonlinear MPC

A non-robust MPC design [Grimm et al., 2004]

- System: $x^+ = \begin{bmatrix} x_1(1-u) \\ |x|u \end{bmatrix} = f(x,u)$
- Input constraints: $\mathbb{U} = [0,1]$
- MPC design: $N = 2$, $\mathbb{X}_f = \{0\}$, $V_f = 0$
- The origin is AS, but **stability has no robustness**

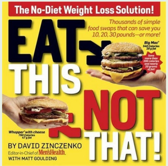


Sufficient conditions for robust nominal stability

- Sufficiently **long prediction horizon** [Grimm et al., 2007]
- **Continuity** of the feasibility region [Pannocchia et al., 2011]

$$\mathcal{U}_N(x) = \{\mathbf{u} \in \mathbb{U}^N \mid \phi(k;x,\mathbf{u}) \in \mathbb{X}, k \in \mathbb{I}_{0:N-1}, \phi(N;x,\mathbf{u}) \in \mathbb{X}_f\}$$

- The above **condition** provides **robustness** also to **suboptimal nonlinear MPC** [Pannocchia et al., 2011]

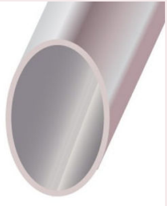


Robust tube-based nonlinear MPC

Same framework as in linear robust MPC [Rawlings and Mayne, 2009]

- **Uncertain nonlinear system**: $x^+ = f(x,u) + w,\ w \in \mathbb{W}$
- **Nominal system**: $z^+ = f(z,v)$
- **Central path** $\mathbb{Z} \subset \mathbb{X}$ and $\mathbb{V} \subset \mathbb{U}$:

$$\bar{\mathbb{P}}_N(z): \qquad \min_{\mathbf{v}} V_N(z,\mathbf{v}) \qquad \text{s.t.} \quad z^+ = f(z,v)$$
$$z(j) \in \mathbb{Z} \qquad \text{for all } j = 0,\ldots,N-1$$
$$v(j) \in \mathbb{V} \qquad \text{for all } j = 0,\ldots,N-1$$
$$z(N) \in \mathbb{Z}_f$$



leading to (an implicit) **nominal control**: $\bar{\kappa}_N(z) = v^0(z;x)$

- **Ancillary controller** (replace $u = v + K(x-z)$)

$$\mathbb{P}_N(x,z): \qquad \min_{\mathbf{v}} V_N(x,z,\mathbf{v}) = \sum_{i=0}^{N-1} \ell(x(i)-z^0(i), u(i)-v^0(i)) \qquad \text{s.t.}$$
$$x^+ = f(x,u), \quad u(i) \in \mathbb{U}, \quad x(N) = z^0(N)$$

The full information estimation problem

Three sets of variables

	System variable	Decision variable	Optimal decision
state	x	χ	$\hat{x}$
process disturbance	w	ω	$\hat{w}$
measurement output	y	η	$\hat{y}$
measurement disturbance	v	ν	$\hat{v}$

Full information objective function

- **True system** evolves as: $x^+ = f(x, w) \qquad y = h(x) + v$
- Given **measurements** $\{y(0), y(1), \ldots, y(T-1)\}$, **cost function** is:

$$V_T(\chi(0), \boldsymbol{\omega}) = \ell_x(\chi(0) - \bar{x}_0) + \sum_{i=0}^{T-1} \ell_i(\omega(i), \nu(i))$$

$$\text{s.t. } \chi^+ = f(\chi, \omega), \quad y = h(\chi) + \nu$$

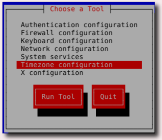
Stability of full information estimator

Optimal full information estimator

- It is the **solution** of

$$\min_{\chi(0),\boldsymbol{\omega}} V_T(\chi(0),\boldsymbol{\omega})$$

- We **denote** the solution as: $\hat{x}(0|T)$, $\hat{w}(i|T)$ for $i = 0,\ldots,T-1$



Global asymptotic stability (GAS)

Definition: Consider the **noise-free case**, i.e. $w(k) = 0$, $v(k) = 0$ for all $k \geq 0$, the estimate is **nominally globally asymptotically stable** if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that for all $(x_0, \bar{x}_0)$ there holds

$$|x(k;x_0) - \hat{x}(k)| \leq \beta(|x_0 - \bar{x}_0|, k) \qquad \text{for all } k \geq 0$$

Result: The estimate obtained from the **full information estimator** is **GAS**

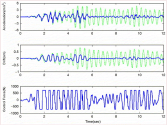
Robust stability of full information estimator

Robust global asymptotic stability (RGAS)

Consider the **noisy case**. The estimate is **robustly** GAS if for all $(x_0, \bar{x}_0)$ and $(\mathbf{w}, \mathbf{v})$ **convergent**, there exist a $\mathcal{KL}$ function $\beta(\cdot)$ and $\mathcal{K}$ functions $\gamma_w(\cdot)$ and $\gamma_v(\cdot)$ such that for all $k \geq 0$:

$$|x(k; x_0) - \hat{x}(k)| \leq \beta(|x_0 - \bar{x}_0|, k) + \gamma_w(\|\mathbf{w}\|) + \gamma_v(\|\mathbf{v}\|)$$

where $\|\mathbf{w}\| = \sup_{k \geq 0} |w(k)|$, $\|\mathbf{v}\| = \sup_{k \geq 0} |v(k)|$



One current limitation

- **Stability** proofs for MHE assume that $(\mathbf{w}, \mathbf{v})$ are **convergent**, i.e.

$$|w(k)| \leq \alpha(\|\mathbf{w}\|, k)$$

- Some recent work [Rawlings and Ji, 2012] **conjectures** that **robust GAS** will hold for simply **bounded** disturbances

Linear state estimation as an optimal control problem

Linear state estimation problem

- **True** system:

$$x^+ = Ax + Gw$$

$$y = Cx + v$$

- **Full information estimator** solves

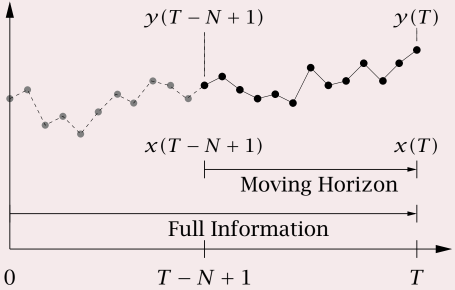
$$\min_{\chi(0),\mathbf{w}} V_T(\chi(0),\mathbf{w}) = |\chi(0) - \bar{x}(0)|^2_{P(0)^{-1}} + \sum_{i=0}^{T-1} |\omega(i)|^2_{Q^{-1}} + |\nu(i)|^2_{R^{-1}}$$

$$\text{s.t. } \chi^+ = A\chi + G\omega, \quad y = C\chi + \nu$$

- Using **forward dynamic programming** we can show that the **full-information estimator** is **equivalent** to the **time-varying Kalman filter**

Moving horizon estimator: introduction

Moving Horizon Estimation: idea and motivation



- At each **new measurement**, the size of the **full information estimation** problem **increases**
- In MHE, the optimal estimation problem has **fixed length** N
- In this way, the **solution time** is bounded

Moving horizon estimator: definitions

General definition

- Given a **prior weighting**, positive definite, function $\Gamma_{T-N}(\cdot)$
- The MHE **objective function** is

$$\hat{V}_T(\chi(T-N),\omega) = \Gamma_{T-N}(\chi(T-N)) + \sum_{i=T-N}^{T-1} \ell_i(\omega(i),\nu(i))$$

$$\text{s.t. } \chi^+ = A\chi + G\omega, \quad y = C\chi + \nu$$

- MHE **solves** a **fixed** and **finite horizon** problem:

$$\min_{\chi(T-N),\omega} \hat{V}_T(\chi(T-N),\omega)$$

- For $k \leq N$, MHE is defined as the **same** as **full information estimator**

The arrival cost

- The term $\Gamma_{T-N}(\chi(T-N))$ is called **arrival cost**
- It takes into account the **past terms**

MHE arrival cost

Zero prior weighting

- One **possible choice** is $\Gamma_{T-N}(p) = 0$
- **Robust** GAS of MHE can be shown for this choice
- However, a **large N** is required to obtain **similar performance** as the **full-information estimator**

Exact arrival cost

- An alternative choice would be to use the **exact arrival cost** of the full-information estimator

$$Z_{T-N}(p) = \min_{\chi(0),\omega} V_{T-N}(\chi(0),\omega) \quad \text{s.t.}$$

$$\chi^+ = f(\chi,\omega), \quad y = h(\chi) + \nu, \quad \chi(T-N) = p$$

- With $\Gamma_{T-N}(\cdot) = Z_{T-N}(\cdot)$, MHE is **identical** to the full-information estimator

Constrained estimation

Constrained full information estimator

- The **constrained full information** estimator solves:

$$\min_{\chi(0),\mathbf{w}} V_T(\chi(0),\mathbf{w}) = \ell_x(\chi(0)-\bar{x}_0) + \sum_{i=0}^{T-1} \ell_i(\omega(i),\nu(i))$$

$$\text{s.t. } \chi^+ = f(\chi,\omega), \quad y = h(\chi)+\nu$$

$$\omega(i)\in\mathbb{W}, \qquad \nu(i)\in\mathbb{V}, \qquad \chi(i)\in\mathbb{X}$$

Constrained MHE

- The **constrained MHE** solves:

$$\min_{\chi(T-N),\mathbf{w}} \hat{V}_T(\chi(T-N),\mathbf{w}) = \Gamma_{T-N}(\chi(T-N)) + \sum_{i=T-N}^{T-1} \ell_i(\omega(i),\nu(i))$$

$$\text{s.t. } \chi^+ = f(\chi,\omega), \quad y = h(\chi)+\nu$$

$$\omega(i)\in\mathbb{W}, \qquad \nu(i)\in\mathbb{V}, \qquad \chi(i)\in\mathbb{X}$$

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