THE LAPLACE METHOD FOR LUMPED PARAMETER MODELS OF CREEP, STRESS RELAXATION AND EPSILON DOT.

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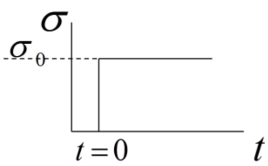
When using Laplace transforms to solve lumped parameter models in the regime of linear viscoelasticity, we exploit the fact that at t=0 both force/stress and deformation are zero, therefore $f(0)$=0.Basically the procedure involves finding the constitutive equation or transfer function of the system in question and the applying the Laplace transform (LT) to it. Then the LT of the stimulus (creep, stress relaxation (SR) or epsilon dot ($\dot{\varepsilon}$)) is applied to the equation. For more details on the epsilon dot ($\dot{\varepsilon}$) method see: **Tirella A, Mattei G, Ahluwalia A. 2014. Strain rate viscoelastic analysis of soft and highly hydrated bio-materials. J Biomed Mater Res Part A 2014:102A:3352–3360.Tirella A, Mattei G, Ahluwalia A. The paper is open access at http://onlinelibrary.wiley.com/doi/10.1002/jbm.a.34914/pdf.**.

1. <u>THE STIMULI : CREEP, SR and</u> $\dot{\varepsilon}$

The basic elements of lumped parameter models are the Hookean spring and the viscous dashpot or piston. Constitutive equations for these two elements are $\sigma(t) = E\varepsilon(t)$ and $\sigma(t) = \eta \dfrac{d\varepsilon}{dt} = \eta\dot{\varepsilon}(t)$ respectively.

For CREEP, given an applied a step stress σ_0 (see Fig 1), one derives the resulting expression for $\varepsilon(t)$. In the case of SR, a step deformation ε_0 (Fig. 2) is applied and the resulting time dependent stress, $\sigma(t)$, is derived. For epsilon dot, the stress, $\sigma(t)$, is expressed in terms of the applied constant deformation rate.

In case of **creep**, a step load or stress, σ_0, is applied at t=0.

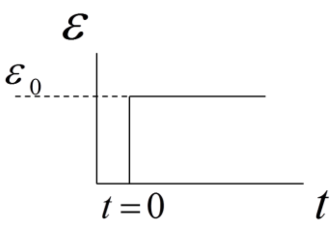


$$\sigma(t) = \sigma_0 H(t)$$

$$L\{\sigma(t)\} = \sigma(s) = \frac{\sigma_0}{s}$$

Fig 1: CREEP

For **SR** we apply a constant deformation, ε_0.

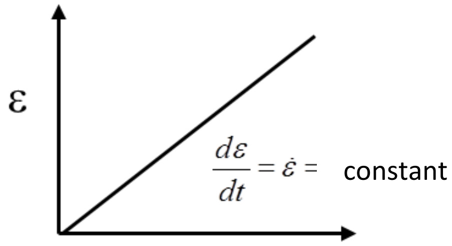


$$\varepsilon(t) = \varepsilon_0 H(t)$$

$$L\{\varepsilon(t)\} = \varepsilon(s) = \frac{\varepsilon_0}{s}$$

Fig 2: SR

Finally, for **epsilon dot ($\dot{\varepsilon}$),** a constant velocity or rate of deformation ($\dot{\varepsilon}$ constant) is applied at t=0.



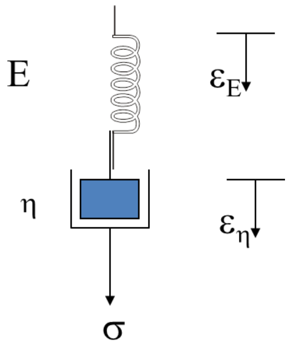
$$\frac{d\varepsilon}{dt} = \dot{\varepsilon} \Rightarrow \varepsilon(t) = \dot{\varepsilon}t$$

$$L\{\varepsilon(t)\} = \varepsilon(s) = \frac{\dot{\varepsilon}}{s^2}$$

Figure 3: Epsilon dot

2. <u>MAXWELL</u>

We write the model equations and their Laplace transforms for the linear Maxwell model:



$$\varepsilon = \varepsilon_\eta + \varepsilon_E \rightarrow \varepsilon(s) = \varepsilon_\eta(s) + \varepsilon_E(s)$$
$$\sigma = \sigma = \sigma$$
$$\sigma = \eta \frac{d\varepsilon_\eta}{dt}, \sigma = E\varepsilon_E \rightarrow \sigma(s) = \eta(s\varepsilon_\eta(s) - \varepsilon_\eta(0))$$
$$\rightarrow \sigma(s) = E\varepsilon_E(s)$$

Figure 4: Maxwell

The constitutive equation is: $\varepsilon = \int \frac{\sigma}{\eta} dt + \frac{\sigma}{E}$

Better written as the time derivative: $\frac{d\varepsilon}{dt} = \frac{\sigma}{\eta} + \frac{1}{E}\frac{d\sigma}{dt}$

2.1 MAXWELL CREEP

The constitutive equation can also be written in the Laplace form.

$$\varepsilon(s) = \frac{\sigma(s)}{s\eta} + \frac{1}{E}\sigma(s)$$
$$\varepsilon(s) = \sigma(s)\left(\frac{1}{s\eta} + \frac{1}{E}\right)$$

Where the transfer function is: $\frac{\varepsilon(s)}{\sigma(s)} = \left(\frac{1}{s\eta} + \frac{1}{E}\right)$ **(Eq. 1)**

We now substitute for the LT of a step load (ie $\sigma(s)=\frac{\sigma_0}{s}$).: $\varepsilon(s)=\sigma_0\left(\frac{1}{s^2\eta}+\frac{1}{sE}\right)$.

And then anti-transform the equation to express *ε(t),* also represented in Fig. 5.

$$\varepsilon(t)=\sigma_0\left(\frac{t}{\eta}+\frac{H(t)}{E}\right)=\sigma_0\left(\frac{t}{\eta}+\frac{1}{E}\right)$$

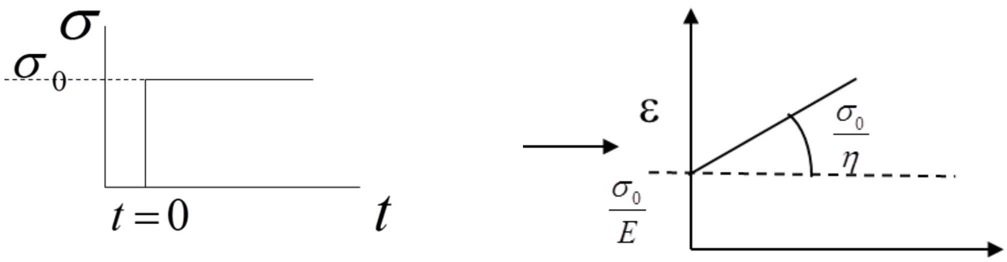


FIG. 5: The creep behaviour of a Maxwell lumped parameter system

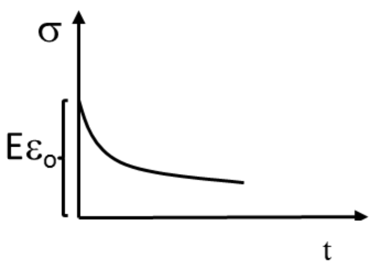
2.2 MAXWELL STRESS RELAXATION

For SR, the same LT of the constitutive equation is expressed as a function of *σ(s)*.

$$\sigma(s)=\varepsilon(s)\left(\frac{s\eta E}{E+s\eta}\right)$$

Whereby applying a step deformation we get: $\sigma(s)=\frac{\varepsilon_0}{s}\left(\frac{s\eta E}{E+s\eta}\right)$

To get the time response, we anti transform the equation, which results in a negative (decaying) exponential, as shown in Fig. 6:



$$\sigma(t)=\varepsilon_0 E e^{-tE/\eta}$$

Fig 6: The SR response for a Maxwell model r

2.3 MAXWELL EPSILON DOT

Here the input is a ramp deformation with $\dot{\varepsilon}$ constant. The Laplace transform of $\dot{\varepsilon}$ constant is $L\{\varepsilon(t)\} = \varepsilon(s) = \frac{\dot{\varepsilon}}{s^2}$

Substituting in the constitutive equation we have

$$\sigma(s) = \varepsilon(s)\left(\frac{s\eta E}{E + s\eta}\right) \to \frac{\dot{\varepsilon}}{s^2}\left(\frac{s\eta E}{E + s\eta}\right) \to \frac{\dot{\varepsilon}\eta E}{sE + s^2\eta} \to \frac{\dot{\varepsilon}E}{s\,E/\eta + s^2}$$

Thus the stress in response to a constant deformation rate is

$$\sigma(t) = \dot{\varepsilon}\eta\left(1 - e^{-tE/\eta}\right)$$

and the behaviour shown in Fig 7.

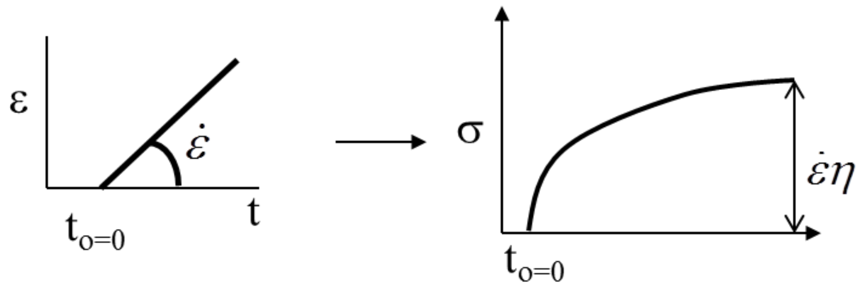


Fig 7: Epsilon dot response of a Maxwell model

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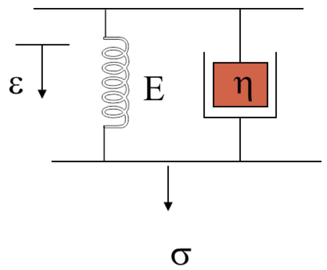
3. <u>VOIGT</u> (note that in some texts this model is referred to as Kelvin or Kelvin-Voigt)

The piston and spring are in parallel, so undergo the same deformation, while the total stress is the sum of the stresses in the two arms (Fig 8).



$$\varepsilon = \varepsilon_\eta = \varepsilon_E$$
$$\sigma = \sigma_\eta + \sigma_E$$
$$\eta\frac{d\varepsilon}{dt} = \sigma_\eta \to \sigma_\eta(s) = \eta(s\varepsilon(s) - \varepsilon(0))$$
$$E\varepsilon = \sigma_E \to \sigma_E(s) = E\varepsilon(s)$$

Fig 8: Viogt model (sometimes also called Kelvin-Voigt)

The constitutive equation is: $\sigma = \eta\frac{d\varepsilon}{dt} + E\varepsilon \to \sigma(s) = \eta s\varepsilon(s) + E\varepsilon(s)$

And the corresponding transfer function is: $\frac{\sigma(s)}{\varepsilon(s)} = \eta s + E$ **(Eq. 2).**

3.1 VOIGT SR

At this point we can substitute the LT of a step deformation ($L\{\varepsilon(t)\} = \frac{\varepsilon_0}{s}$) into the constitutive equation to obtain:

$$\sigma(s) = \frac{\varepsilon_0}{s}(\eta s + E) \rightarrow \sigma(s) = \varepsilon_0 \eta + \frac{\varepsilon_0 E}{s}$$

Which can be easily anti-transformed to give: $\sigma(t) = \varepsilon_0 \eta . \delta(t) + \varepsilon_0 E$

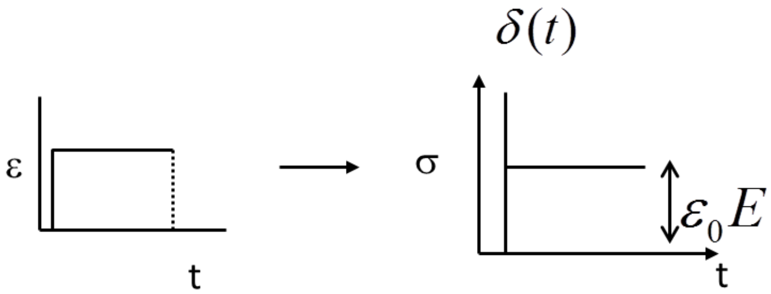


Fig 9: The SR of a Voigt model

3.2 VOIGT CREEP

Once again, a step stress is imposed as an input stimulus to the transfer function.

$$\frac{\varepsilon(s)}{\sigma(s)} = \frac{1}{\eta s + E} \rightarrow \varepsilon(s) = \frac{\sigma_0}{s} \frac{1}{\eta s + E} \rightarrow \sigma_0 \frac{1}{s^2 \eta + sE}$$

The anti-transform is $\varepsilon(t) = \frac{\sigma_0}{E}(1 - e^{-tE/\eta})$, with response reported in Fig. 10.

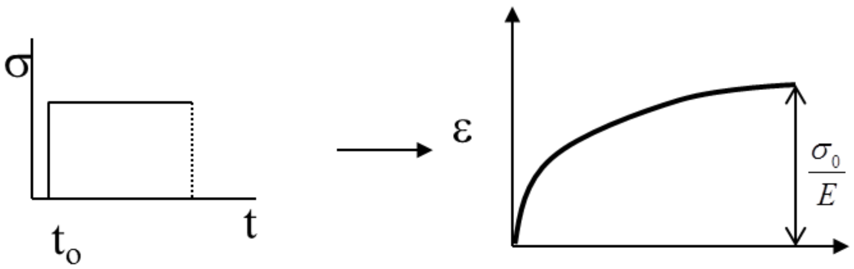


Fig. 10: Creep response of a Voigt system

3.3 VOIGT EPSILON DOT

Here the input is a ramp deformation with $\dot{\varepsilon}$ constant. The Laplace transform of $\dot{\varepsilon}$ constant is $L\{\varepsilon(t)\} = \varepsilon(s) = \frac{\dot{\varepsilon}}{s^2}$ (see section 1).

Substituting in the constitutive equation (Eq 2) we have

$$\sigma(s) = \varepsilon(s)(\eta s + E) \rightarrow \frac{\dot{\varepsilon}}{s^2}(\eta s + E) \rightarrow \dot{\varepsilon}\left(\frac{\eta}{s} + \frac{E}{s^2}\right)$$

Thus the stress response to a constant deformation rate is $\sigma(t) = \dot{\varepsilon}(\eta + Et)$, i.e. the stress increases linearly with time as shown in Figure 11.

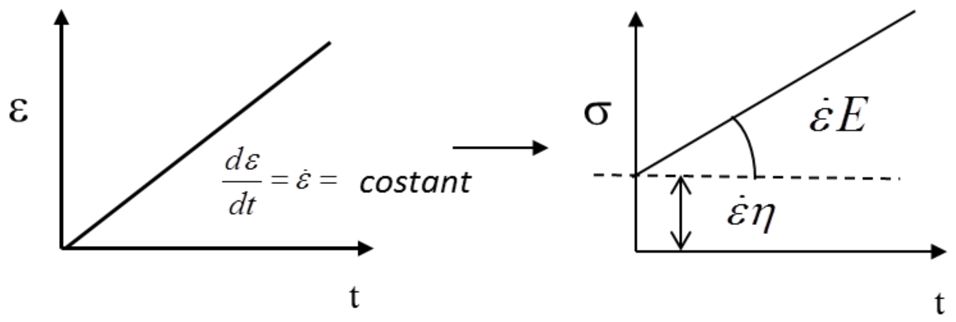


Fig. 11: The Voigt lumped parameter model response to a constant deformation rate (epsilon dot)

4. KELVIN or SLS

For the Kelvin or Standard Linear Solid Model, there are two different but equivalent configurations. Although either can be used, for ease of calculation the 3 element Generalized Maxwell model (GM) is used for SR and epsilon dot, while the SLS Kelvin form of the model is often used in the case of creep. Unfortunately these models do not have a unique naming convention.

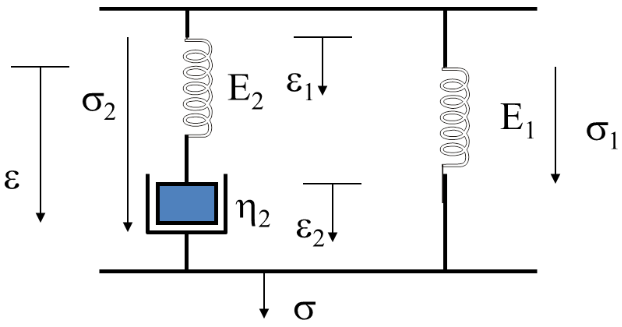


Fig. 12a: The SLS parallel (SLS//) model or 3 element Generalized Maxwell model.

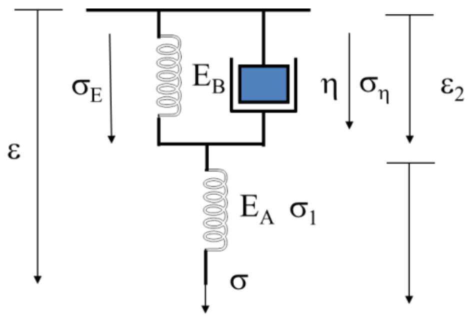


Fig. 12b:The SLS series (SLS ⊥) or Kelvin model.

The two models are equivalent, and this can be demonstrated by inspecting their instantaneous and equilibrium responses. A simple way of analysing the lumped parameters is to think of the piston as being "shorted" at the instant a stimulus is applied, and being an "open" or infinite resistance (to force or deformation) at equilibrium (or t→ ∞). The table below shows the equivalencies (Table 1).

	SLS//	**SLS ⊥**
Instantaneous (E_{inst})	$E_1 + E_2$	E_A
Equilibrium (E_∞)	E_1	$\frac{E_A E_B}{E_A + E_B}$

4.1 CONSTITUTIVE EQUATIONS & Transfer functions SLS//

In general it is easier to resolve the SR and epsilon dot response for the SLS// and the CREEP response for the SLS ⊥. The method is always the same: express *ε(s)* and *σ(s)* in terms of the lumped parameters, eliminating the intermediate stresses and deformations. Then express the stress or deformation in partial fractions which can be suitably anti-transformed to the time domain.

For example, the 5 equations and their LTs describing this system are:

$$\sigma_1 = \varepsilon E_1 \rightarrow \sigma_1(s) = E_1\varepsilon(s)$$
$$\varepsilon = \varepsilon_1 + \varepsilon_2 \rightarrow \varepsilon(s) = \varepsilon_1(s) + \varepsilon_2(s)$$
$$\sigma_2 = \varepsilon_1 E_2 \rightarrow \sigma_2(s) = \varepsilon_1(s)E_2$$
$$\sigma_2 = \eta_2 \frac{d\varepsilon_2}{dt} \rightarrow \sigma_2(s) = s\eta_2\varepsilon_2(s)$$
$$\sigma = \sigma_1 + \sigma_2 \rightarrow \sigma(s) = \sigma_1(s) + \sigma_2(s)$$

So the constitutive equation (obtained by eliminating intermediate stresses and strains) in the Laplace domain is

$$\sigma(s) = \varepsilon(s)\left(E_1 + \frac{s\eta_2 E_2}{s\eta_2 + E_2}\right)$$

And the transfer function is simply

$$\frac{\sigma(s)}{\varepsilon(s)} = E_1 + \frac{s\eta_2 E_2}{s\eta_2 + E_2}$$ **(Eq. 3)**

From this it is easy to show that were we to add successive pistons and springs in parallel (Fig 13) the transfer function will include further terms which are added on in a series fashion. This type of model is known as the Generalised Maxwell model (or Generalised SLS).

$$\frac{\sigma(s)}{\varepsilon(s)} = E_1 + \frac{s\eta_2 E_2}{s\eta_2 + E_2} + \frac{s\eta_3 E_3}{s\eta_3 + E_3} + \ldots\ldots \frac{s\eta_n E_n}{s\eta_n + E_n}$$

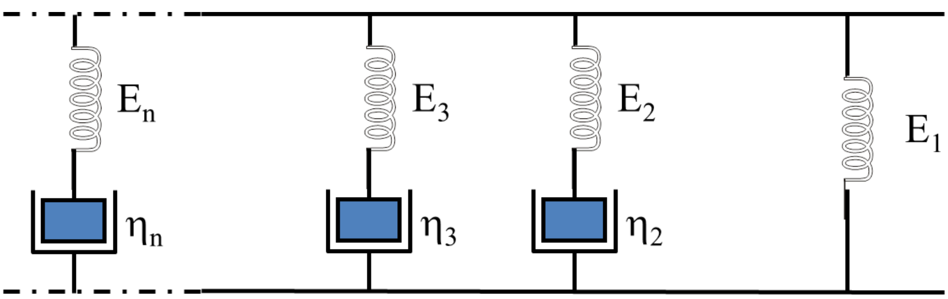


Fig. 13: Generalised parallel SLS (or Generalised Maxwell) model

4.2 SR for the SLS//

We use Eq. 3 derived above and substitute for the LT of a step deformation, $\frac{\varepsilon_0}{s}$.

$$\sigma(s) = \frac{\varepsilon_0}{s}\left(E_1 + \frac{s\eta_2 E_2}{s\eta_2 + E_2}\right)$$

Antitransforming we get

$$\sigma(t) = \varepsilon_0 \left(E_1 + E_2 e^{-\left(\frac{E_2}{\eta_2}\right)t} \right)$$

Note that successive series of pistons and springs add another exponential term to the equation.

The response is reported in Figure 13.

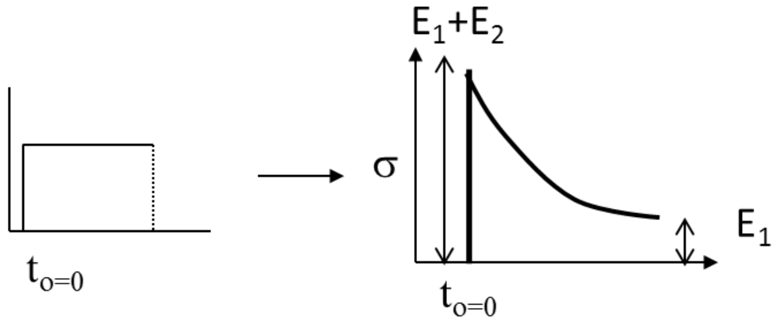


Fig. 13: SR of a 3 element SLS

4.3 EPSILON DOT for SLS//

For the epsilon dot response, substituting $\varepsilon(s)$ for $\frac{\dot{\varepsilon}}{s^2}$ gives

$$\sigma(s) = \frac{\dot{\varepsilon}}{s^2} E_1 + \frac{\dot{\varepsilon}\eta_2 E_2}{s^2\eta_2 + sE_2}$$

Which can be antitransformed to give the response illustrated in Figure 14 (equation below).

$$\sigma(t) = \dot{\varepsilon} E_1 t + \dot{\varepsilon}\eta_2 \left(1 - e^{-\left(\frac{E_2}{\eta_2}\right)t} \right)$$

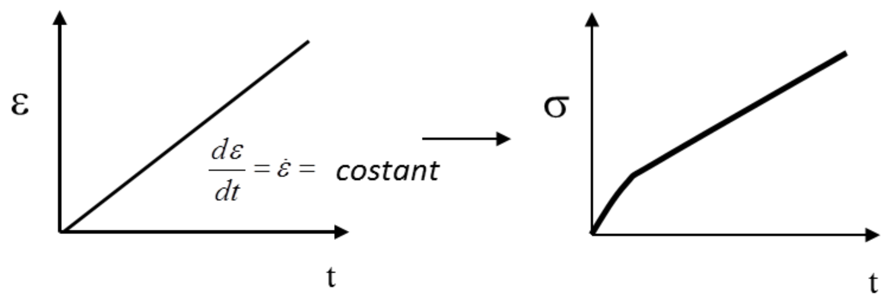


Fig. 14: The $\dot{\varepsilon}$ response for the SLS// model

4.4 CREEP for the SLS//

The reciprocal of the transfer function (Eq. 3) is expressed as

$$\frac{\varepsilon(s)}{\sigma(s)} = \left(\frac{s\eta_2 + E_2}{E_1E_2 + s\left(E_1\eta_2 + E_2\eta_2\right)} \right)$$

Then, substituting the LT of a creep stimulus for $\sigma(s)$ ($\sigma(s) = \frac{\sigma_0}{s}$) gives the deformation in the Laplace domain, which can be expressed as the sum of two fractions, and the instantaneous elastic modulus (see Table 1).

$$\varepsilon(s) = \sigma_o \left(\frac{\eta_2}{E_1E_2 + s\left(E_1\eta_2 + E_2\eta_2\right)} + \frac{E_2}{sE_1E_2 + s^2\left(E_1\eta_2 + E_2\eta_2\right)} \right) =$$

$$\sigma_o \left(\frac{1/(E_1+E_2)}{\frac{E_1E_2}{\eta_2(E_1+E_2)} + s} + \frac{E_2/\eta_2(E_1+E_2)}{s\left(\frac{E_1E_2}{\eta_2(E_1+E_2)} + s\right)} \right) = \sigma_o \left(\frac{1/(E_1+E_2)}{\frac{E_{series}}{\eta_2} + s} + \frac{E_2/\eta_2(E_1+E_2)}{s\left(\frac{E_{series}}{\eta_2} + s\right)} \right)$$

Anti transforming gives

$$\varepsilon(t) = \sigma_o \left(\frac{1}{\left(E_1+E_2\right)} e^{-\frac{Eseries}{\eta_2}t} + \frac{1}{E_1}\left(1 - e^{-\frac{Eseries}{\eta_2}t}\right) \right)$$

$$= \sigma_o \left(\frac{1}{E_1} - e^{-\frac{Eseries}{\eta_2}t} \left(\frac{E_2}{E_1\left(E_1+E_2\right)} \right) \right)$$

Figure 15 illustrates the creep response.

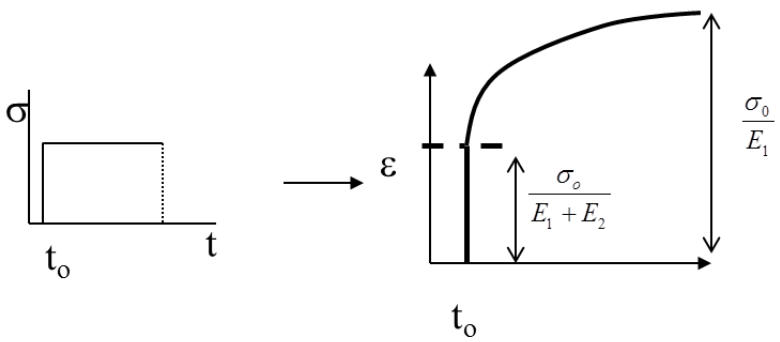


Figure 15: Creep for the SLS// model

The above expressions can be verified at t=0 and t=∞. Resultant elastic moduli should correspond to the values in table 1.

Finally, the derivation of CREEP, SR and EPSILON DOT for the SLS⊥ model is left as an exercise.

LAPLACE TRANSFORMS used here (t>0).

f(t)	*F(s)*
$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\dot{f}$	$sf(s) - f(0)$
e^{-at}	$\frac{1}{s+a}$
$\frac{1-e^{-at}}{a}$	$\frac{1}{s(s+a)}$
$\delta(t)$	*1*

A similar approach can be used for stimuli which:

a) Begin at t=0+T
b) Relaxation responses after the CREEP and SR the stimulus is removed (i.e. respectively stress and strain return to zero as a negative step).